VIRTUAL MASSES OF SOME CURVILINEAR CONTOURS IMMERSED IN DETACHED FLOW

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1. An exact solution in closed form of the problem of the detached flow past one of the family of curves $L(m, \nu)$ was given by Pykhteev [1]. In this article we consider the impact upon the same family of curves of the detached streamlined flow of indefinite extent of an ideal incompressible fluid.

The curve $L(m, \nu)$ which is a function of two parameters m and ν is assumed to be in the plane of the flow $z = x + iy$. It is given in the form of the parametric equation

$$x = -\lambda \frac{\cos^m \nu}{m} \operatorname{ctg} \nu \int_{^1/_2 \pi}^{\vartheta} \frac{1 + \sqrt{1 - \operatorname{ctg}^2 \nu \operatorname{tg}^2 \theta}}{[1 + \sqrt{1 - \operatorname{ctg}^2 \nu \operatorname{tg}^2 \theta} \sin \nu]^m} \frac{\cos \vartheta \, d\vartheta}{\cos^{m+2} \theta}$$

$$\left(\frac{\pi - 2\vartheta}{2m} = \theta\right) \qquad (1.1)$$

$$y = -\lambda \frac{\cos^m \nu}{m} \operatorname{ctg} \nu \int_{^1/_2 \pi}^{\vartheta} \frac{1 + \sqrt{1 - \operatorname{ctg}^2 \nu \operatorname{tg}^2 \theta}}{[1 + \sqrt{1 - \operatorname{ctg}^2 \nu \operatorname{tg}^2 \theta} \sin \nu]^m} \frac{\sin \vartheta \, d\vartheta}{\cos^{m+2} \theta}$$

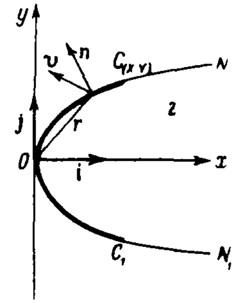

Fig. 1.

where θ is the angle of the tangent line to the x-axis, which varies between $0 \leqslant \pi/2 - \theta \leqslant m\nu$ where $0 \leqslant \nu < \pi/2m$. The curve $L(m, \nu)$ with $\nu \neq 0$ is increasing monotonically. It is symmetrical with respect to the x-axis, passes through the origin of the coordinate system, and is tangent to y-axis. When $\nu = 0$ the curve $L(m, \nu)$ becomes a segment of a straight line, coincident with the y-axis (Fig. 1). At the end point of the curve $C(X, Y)$ the angle of inclination of the tangent line is $\theta = \pi/2 - m\nu$, and the length of the arc is S. For every curve $L(m, \nu)$ the parameter λ in equation (1.1) is determined when one of the three quantities X, Y or S

is given. The integrals in (1.1) for all integral and for many fractional values of m are expressible in terms of elementary or tabulated functions. It is easy to determine the curvature of the curve:

$$K(\vartheta) = \frac{d\vartheta}{ds} = -\frac{m \operatorname{tg} \nu}{\lambda \cos^m \nu} \frac{[1 + \sqrt{1 - \operatorname{ctg}^2 \nu \operatorname{tg}^2 \theta} \sin \nu]^m}{1 + \sqrt{1 - \operatorname{ctg}^2 \nu \operatorname{tg}^2 \theta}} \cos^{m+2} \theta \qquad (1.2)$$

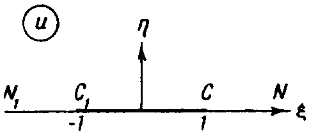

Fig. 2.

2. Let the curve $L(m, \nu)$, which forms a boundary of a detached flow, suddenly acquire the translational velocity $\mathbf{v}(v_2, v_1)$ and the rotational velocity ω with respect to the point O. The resulting additional (impulsive) flow has a complex velocity potential $w = \phi + i\psi$, where ϕ depends upon the density of the fluid ρ and the impulsive pressure p through the relationship $p = -\rho\phi$. The harmonic function $\phi(x, y)$ satisfies the boundary conditions: at the free surfaces $p = 0$, consequently, $\phi = 0$; at the curve $L(m, \nu)$ the normal component of the velocity $\partial\phi/\partial n = v_n$ is knownk where in the case under consideration

$$v_n = \mathbf{n} \cdot \mathbf{v} + \mathbf{n}\omega\mathbf{r} = v_1 \cos\vartheta - v_2 \sin\vartheta + \omega(x\cos\vartheta + y\sin\vartheta) \qquad (2.1)$$

Furthermore, from physical considerations it follows that the complex velocity of the impulsive flow dw/dz becomes infinite at the ends of the curve and is equal to zero in the stream at infinity.

We shall transform the region of the flow of the plane (z) conformally into the upper half-plane of the parametric variable $u = \xi + i\eta$ (Fig. 2). The corresponding points in Figs. 1 and 2 are denoted by the same letters. The transformation is derived from the paper [1] and in our notation it is written

$$\frac{dz}{du} = \lambda i (1 + \sqrt{1 - u^2}) \left[\frac{1 - (\sqrt{1 - u^2} + iu) \operatorname{tg} {}^1\!/_2 \nu}{1 + (\sqrt{1 - u^2} + iu) \operatorname{tg} {}^1\!/_2 \nu}\right]^m \qquad (2.2)$$

The boundary conditions, expressed in terms of the function dw/du, in the plane u correspondingly will be in the form:

$$\operatorname{Re} \frac{dw}{du} = 0, \qquad |\xi| > 1, \qquad \eta = 0 \qquad (2.3)$$

$$\operatorname{Im} \frac{dw}{du} = -v_n \left|\frac{dz}{du}\right|, \qquad |\xi| < 1, \qquad \eta = 0 \qquad (2.4)$$

Furthermore, dw/du must at the ends of the contour $u = \pm 1$ become infinite of the order minus one-half, and at infinity it must be zero not lower than of the second order. The analytical function dw/du in the upper half-plane (u) is found by the methods of the theory of thin wing and of the impact upon the incompressible fluid [2] :

$$\frac{dw}{du} = \frac{1}{\pi i \sqrt{u^2 - 1}} \int_{-1}^{1} \left[v_n(u) \left| \frac{dz}{du} \right| \right]_{u=\xi} \frac{\sqrt{1-\xi^2}}{\xi - u} d\xi \qquad (2.5)$$

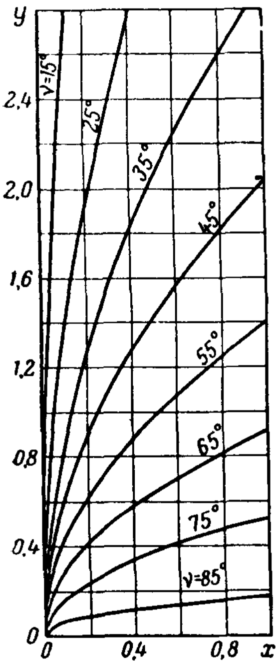

Fig. 3.

where $|dz/du|$ and θ is determined from (2.2) for $-1 < u < 1$:

$$\left| \frac{dz}{du} \right| = \lambda (1 + \sqrt{1-u^2}) \left[\frac{1 - \sqrt{1-u^2} \sin \nu}{1 + \sqrt{1-u^2} \sin \nu} \right]^{1/2 m} \qquad (2.6)$$

$$\vartheta = \arg \frac{dz}{du} = \frac{1}{2} \pi - m \operatorname{arc\,tg} (u \operatorname{tg} \nu) \qquad (2.7)$$

and $x(u)$, $y(u)$ are found from (1.1) and (2.7). In such a manner, the formulas (2.5) and (1.1), (2.1), (2.6), (2.7) yield the general solution of the problem.

Making use of the fact that v_n is expressible linearly in terms of v_1, v_2 and ω it is convenient to express the desired potential in the form

$$w = v_1 w_1 + v_2 w_2 + \omega w_3 \qquad (w_k = \varphi_k + i\psi_k) \qquad (2.8)$$

where w_1, w_2, w_3 are the complex potentials of the vertical, horizontal and the rotational impacts with unit velocities.

3. At the time of the impact on the element ds of the curve there are acting the elemental impulse $d\mathbf{J}$ and the moment $d\mathbf{M}$ with respect to the point O (Fig. 1):

$$d\mathbf{J} = p\, ds\, \mathbf{n} = -\mathbf{i}\, p\, ds \sin \vartheta + \mathbf{j}\, p\, ds \cos \vartheta = -\mathbf{i}\, p\, dy + \mathbf{j}\, p\, dx$$

$$dM = |\mathbf{r} \times p\, ds\, \mathbf{n}| = p\, ds\, (x \cos \vartheta + y \sin \vartheta) = p\, (x\, dx + y\, dy) = \frac{1}{2} p\, d\, |z|^2$$

When summing up along the contour and noting that $p = -\rho\varphi$, we obtain $\mathbf{J}(J_x, J_y)$ and M:

$$J_x = \rho \int_{C_1C} \varphi\, dy, \qquad J_y = -\rho \int_{C_1C} \varphi\, dx, \qquad M = -\frac{1}{2} \rho \int_{C_1C} \varphi\, d\, |z|^2$$

We shall now consider the variable u. When integrating by parts and taking into account that at the boundaries of the contour $\varphi = 0$, we have

$$J_x = -\rho \int_{-1}^{1} y(u) \frac{d\varphi}{du} du, \qquad J_y = \rho \int_{-1}^{1} x(u) \frac{d\varphi}{du} du, \qquad M = \frac{1}{2} \rho \int_{-1}^{1} |z|^2 \frac{d\varphi}{du} du \quad (3.1)$$

The quantities $-J_x$, $-J_y$, $-M$ may be considered as the momentum and the moment of momentum of the fluid and be represented in the form

$$\begin{aligned} -J_y &= \mu_{11}v_1 + \mu_{12}v_2 + \mu_{13}\omega \\ -J_x &= \mu_{21}v_1 + \mu_{22}v_2 + \mu_{23}\omega \\ -M &= \mu_{31}v_1 + \mu_{32}v_2 + \mu_{33}\omega \end{aligned} \tag{3.2}$$

where μ_{ik} are the coefficients of virtual mass.

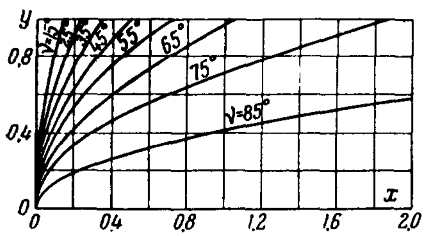

Fig. 4.

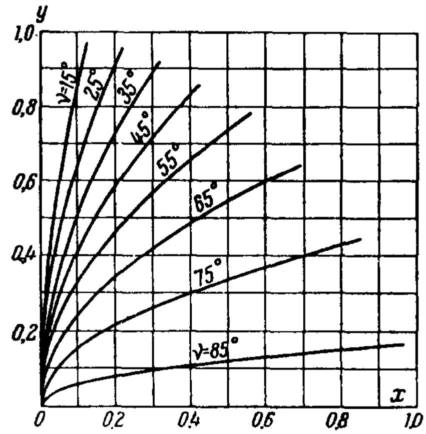

Fig. 5.

Comparing (3.2) and (3.1) and taking into account (2.8) we obtain the computation formulas

$$\mu_{1k} = -\rho\int_{-1}^{1} x(u)\,\frac{d\varphi_k}{du}\,du \qquad \mu_{2k} = \rho\int_{-1}^{1} y(u)\,\frac{d\varphi_k}{du}\,du, \qquad \mu_{3k} = -\frac{1}{2}\rho\int_{-1}^{1} |z|^2\,\frac{d\varphi_k}{du}\,du \tag{3.3}$$

Because of the symmetry $\mu_{12} = \mu_{21} = \mu_{23} = \mu_{32} = 0$. The remaining coefficients are different from zero.

As an example we shall give the calculated μ_{ik} for the curves of the class $L(1, \nu)$. Assuming $n = 1$ in the formulas (2.6), (2.7), according to (2.1), (2.5) and (2.8) we find

$$\frac{dw_1}{du} = -\frac{\lambda\csc\nu}{\pi(\operatorname{ctg}^2\nu + u^2)\sqrt{1-u^2}}\Big\{A(\nu) + \Big[\tfrac{1}{2}\pi\sin\nu - (\pi+2)(1-\sin\nu)\Big]u^2 - $$
$$- u^4\pi\sin\nu + (1-\sin\nu)\,u\,(1-u^2)\ln\frac{u-1}{u+1} + i\pi(1-\sin\nu + u^2\sin\nu)\,u\sqrt{1-u^2}\Big\} \tag{3.4}$$

$$\frac{dw_2}{du} = -\frac{\lambda\cos\nu\csc^2\nu}{\pi(\operatorname{ctg}^2\nu + u^2)\sqrt{1-u^2}}\Big\{uB(\nu) - u^3\pi\sin\nu + (1-\sin\nu)(1-u^2)\ln\frac{u-1}{u+1} + $$
$$+ i\pi(1-\sin\nu + u^2\sin\nu)\sqrt{1-u^2}\Big\} \tag{3.5}$$

$$A(\nu) = 2 \operatorname{ctg}\nu\,(1-\sin\nu)\,(\nu\csc^2\nu - \operatorname{ctg}\nu) + \pi \operatorname{ctg}^2\nu \left[\tfrac{1}{2}\sin\nu - (\sec\nu - 1)(\operatorname{cosec}\nu - 1)\right]$$

$$B(\nu) = \pi \sec\nu\,(\csc\nu - 1)\left(1 - \tfrac{1}{2}\sin 2\nu - \cos\nu - \tfrac{2\nu}{\pi}\right)$$

It is easy to verify that dw_1/du, dw_2/du satisfy the boundary conditions (2.3), (2.4) and also the additional requirements of a physical nature.

Assuming $m = 1$ in (1.1), (2.7) we find the equation $L(1, \nu)$:

$$x(u) = \lambda(1-\csc\nu)\left[\frac{u^2}{2(1-\csc\nu)} + 1 - \sqrt{1-u^2} + \csc\nu \ln\frac{1+\sqrt{1-u^2}\sin\nu}{1+\sin\nu}\right]$$

$$y(u) = \lambda \operatorname{ctg}\nu\,(\csc\nu - 1)\left[\frac{u}{\csc\nu - 1} - \arcsin u + 2\sec\nu \operatorname{arc\,tg}\left(\frac{1-\sqrt{1-u^2}}{u}\operatorname{tg}\frac{\pi - 2\nu}{4}\right)\right] \tag{3.6}$$

The function dw_3/du is not expressible in terms of elementary functions and may be found approximately by the use of (2.5), if we expand $x(u)$ and $y(u)$ in (2.1), for example, into the series of Legendre polynomials. For $u = 1$ the formulas (3.6) give X, Y of the end of the curve:

$$X = \lambda(1-\csc\nu)\left[\frac{1}{2(1-\csc\nu)} + 1 - \csc\nu \ln(1+\sin\nu)\right]$$

$$Y = \lambda \operatorname{ctg}\nu\,(\csc\nu - 1)\left[\frac{1}{\csc\nu - 1} - \frac{\pi}{2} + \left(\frac{\pi}{2} - \nu\right)\sec\nu\right] \tag{3.7}$$

Finally, integrating (2.6) from 0 to 1 for $m = 1$, we obtain:

$$S = \lambda\frac{1-\sin\nu}{\sin^2\nu}\left[F\left(\sin\nu, \frac{\pi}{2}\right) - E\left(\sin\nu, \frac{\pi}{2}\right) + \frac{1}{2(\csc\nu - 1)} - \frac{1-\sin\nu}{2}\ln\operatorname{ctg}\frac{\pi - 2\nu}{4}\right] \tag{3.8}$$

where F and E are complete elliptical integrals of the first and second orders.

Figures 3, 4, and 5 represent correspondingly the families of the curves

$$X = 1\left(\frac{x(u)}{X}, \frac{y(u)}{X}\right), \qquad Y = 1\left(\frac{x(u)}{Y}, \frac{y(u)}{Y}\right), \qquad S = 1\left(\frac{x(u)}{S}, \frac{y(u)}{S}\right)$$

In Figs. 6, 7 and 8 are given the non-dimensional coefficients of the virtual masses as functions of ν or of the non-dimensional maximum curvature of the curve $K(\pi/2)X$, $K(\pi/2)Y$, $K(\pi/2)S$ according to (3.3)-(3.8) and (1.2). As seen from Figs. 6 and 7, the coefficients $\mu_{11}/\rho X^2$, $\mu_{22}/\rho Y^2$ practically do not depend upon the curvature of the curves of their corresponding families $X = 1$ and $Y = 1$, i.e. they are determined only by the magnitude of the projection of the maximum cross-section of the curve perpendicular to the direction of its impact. This fact serves to explain that $\lim \mu_{22}/\rho Y^2 = \pi/2$ for $\nu \to \pi/2$, a finite number.

For $\nu \to 0$ the curves $Y = 1$ and $S = 1$ represent a vertical plate of

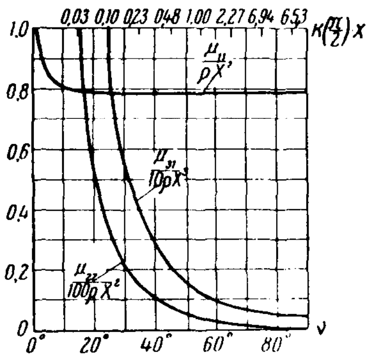

Fig. 6.

length equal to two, placed into a streamlined detached flow. In this case, equation (3.3) yields values already known [3].

$$\frac{\mu_{22}}{\rho Y^2} = \frac{\mu_{22}}{\rho S^2} = 1.6896$$

For $\nu \to \pi/2$ the curves $X = 1$ and $S = 1$ represent a plate of unit leggth length bounding streamlined flow on both sides; for this plate according to equation (3.3) we have [2]

$$\frac{\mu_{11}}{\rho X^2} = \frac{\mu_{11}}{\rho S^2} = \frac{\pi}{4}, \quad \frac{\mu_{31}}{\rho X^3} = \frac{\mu_{31}}{\rho S^3} = \frac{\pi}{8}, \quad \frac{\mu_{33}}{\rho X^4} = \frac{\mu_{33}}{\rho S^4} = \frac{9\pi}{128}.$$

In conclusion we shall note, that in the case of a horizontal impact by the curve $L(1, \nu)$, as opposed to a similar impact by the wedge [4], the sign of the tangential velocity component of the impulsive flow is not changed along the curve when going from the point O to C (Fig. 1). For the vertical impact by the curve $L(1, \nu)$ the sign does change.

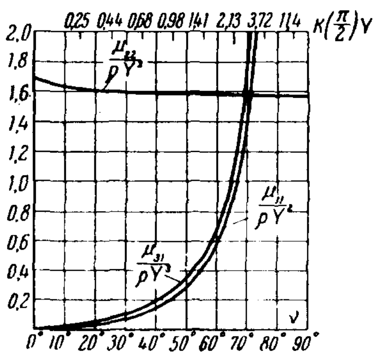

Fig. 7.

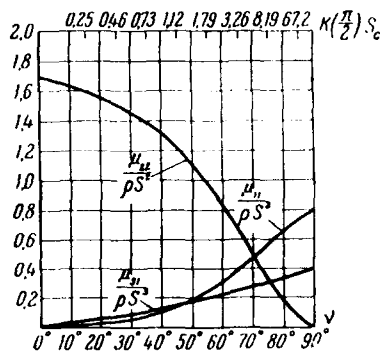

Fig. 8.

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